PAPER - 10

STATICS

OBJECTIVES

This course introduces the students the basic concepts of forces, moments, couple, friction law virtual displacement and work, catenary and the centre of gravity and kinematics. This course stresses the development of skills in formation of suitable mathematical models and problems solving techniques.

UNIT- I

Forces, Type of forces- Resultant of three forces related to triangle acting at a point - Resultant of several forces acting on a particle - Equilibrium of a particle under three forces - Equilibrium of a particle under several forces - Limiting Equilibrium of a particle on an inclined plane.

UNIT- II

Moment of a forces- General motion of a Rigid body- Equivalent system of forces – Parallel forces- Forces along the sides of the triangle.

UNIT- III

Couples- Resultant of several coplanar forces – Equation of line of action of the resultant – Equilibrium of a rigid body under three coplanar forces.

UNIT - IV

Reduction of coplanar forces into a force and a couple – Friction – laws of friction – cone of friction and angle of friction – Applications involving frictional forces.

UNIT - V

Center of mass – Center of mass of a triangular lamina – Three particles of same mass - Three particles of certain masses – uniform rods forming a triangle – lamina in the form of a trapezium and solid tetrahedron – Center of mass using integration – circular arc – circular lamina – elliptic lamina – solid hemisphere – solid right circular cone – hemispherical shell – hollow right circular cone.

Recommended Text

P. Duraipandian, LaxmiDuraipandian ,MuthamizhJayapragasam, Mechanics, 6-e,
S. Chand and Company Ltd, 2005**.**

Reference Books

1. S. Narayanan, R. HanumanthaRao, K. Sitaraman, P. Kandaswamy, *Statics*, S. Chand and Company Ltd, New Delhi.
2. S. L. Loney, *An Elementary Treatise on Statics*, Combridge University Press, 1951
3. A.V. Dharmapadam(1991) *Mechanics*. S. Viswanathan Printers & Publishers. Chennai.
4. M.K. Venkataraman (1990) *Statics*. A Rajhans Publications. (16^{th}Edn), Meerut.
5. Joseph F. Shelley (2005) *Vector Mechanics for Engineers Vol-I: Statics*, Tata McGraw Hill Edition, New Delhi.

<u>UNIT - 1</u>

FORCE

Definition:

A Force is any interaction that, when unopposed will change the motion of an object.

TYPES OF FORCES

- Earth Gravitation
- Tension
- Reaction and
- Resistance

RESULTANT FORCE

Let $\vec{F}_1$ and $\vec{F}_2$ be two forces acting on a particle then $\vec{F}_1 + \vec{F}_2$ is called resultant force of $\vec{F}_1$ and $\vec{F}_2$.

NEWTON'S LAW OF MOTION

FIRST LAW

A Particle remains at rest or in a state of uniform motion in a straight line unless acted on by on impressed force

SECOND LAW

The Rate of change of momentum of a particle is proportional to the impressed force and it is in the direction of force.

NOTE

LINEAR MOMENT

Let m be the mass of a particle and $\vec{V}$ be the velocity then $\overrightarrow{mV}$ is called linear momentum or momentum.

By 2^{nd} Law

Rate of change of momentum is proportional to impressed force

$$ie)\ \frac{d}{dt}\left(m\dot{\vec{r}}\right)\ \alpha\ \vec{F}$$

$$m\ddot{\vec{r}} = K\ \vec{F}$$

$$m\vec{a} = K\vec{F}$$

For a unit force

For $F = 1,\ m = 1,\ a = 1$

$$\Rightarrow K = 1$$

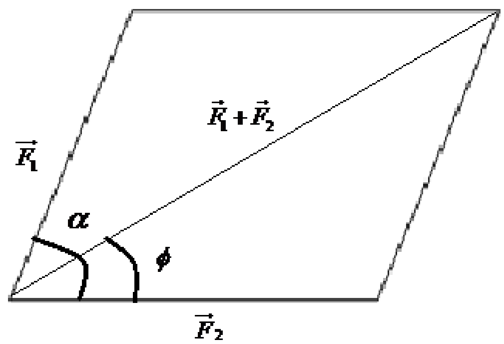
$$\therefore m\vec{a} = \vec{F}$$

THIRD LAW

To every action there is an equal and opposite reaction.

Problem: 1

To find the magnitude and direction of the resultant force of $\vec{F_1}$ and $\vec{F}_2$



Let $\vec{F_1}$ and $\vec{F}_2$ be two forces of a particle and $\vec{F_1}+\vec{F}_2$ be its resultant force.

Let α be the angle between $\vec{F_1}$ and $\vec{F}_2$

Magnitude Resultant Force $= \left|\vec{F_1}+\vec{F_2}\right|$

$$= \sqrt{\left(\vec{F_1}+\vec{F_2}\right)\bullet\left(\vec{F_1}+\vec{F_2}\right)}$$

$$= \sqrt{\vec{F_1}\bullet\vec{F_1}+\vec{F_1}\bullet\vec{F_2}+\vec{F_2}\bullet\vec{F_1}+\vec{F_2}\bullet\vec{F_2}}$$

$$= \sqrt{\vec{F_1}^2+\overrightarrow{2F_1}\bullet\vec{F_2}+\vec{F_2}^2} \text{ since } \vec{a}\bullet\vec{b} = |\vec{a}|\bullet|\vec{b}|\cos\phi$$

Here $\left|\vec{F_1}+\vec{F_2}\right| = \sqrt{\vec{F_1}^2+\vec{F_2}^2+\overrightarrow{2F_1}\bullet\vec{F_2}\cos\alpha}$

(OR)

$$\left|\vec{F_1}+\vec{F_2}\right|^2 = \vec{F_1}^2+\vec{F_2}^2+\overrightarrow{2F_1}\bullet\vec{F_2}\cos\alpha$$

Let ϕ be the angle between $\vec{F_1}$ and $\vec{F_1}+\vec{F_2}$

$$\tan\phi = \frac{\left|\vec{a}X\vec{b}\right|}{\vec{a}\bullet\vec{b}} \ (\,formula)$$

$$= \frac{\left|\vec{F_1} X \left(\vec{F_1} + \vec{F_2}\right)\right|}{\vec{F_1} \bullet \left(\vec{F_1} + \vec{F_2}\right)}$$

$$= \frac{\left|\left(\vec{F_1} X \vec{F_1}\right) + \left(\vec{F_1} X \vec{F_2}\right)\right|}{\left(\vec{F_1} \bullet \vec{F_1}\right) + \left(\vec{F_1} \bullet \vec{F_2}\right)} \quad \text{since } \vec{a} X \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \sin\phi$$

$$= \frac{\left|0 + \left(F_1 F_2 \sin\alpha\right)\right|}{F_1 + \left(F_1 F_2 \cos\alpha\right)}$$

$$\tan\phi = \frac{F_1 F_2 \sin\alpha}{F_1\left(F_1 + F_2 \cos\alpha\right)}$$

$$\phi = \tan^{-1}\left(\frac{F_2 \sin\alpha}{\left(F_1 + F_2 \cos\alpha\right)}\right)$$

Corollary:

Suppose $\vec{F_1}$ and $\vec{F_2}$ have same magnitude (say F) then

Put $F_1 = F_2 = F$

$$\left|\vec{F_1} + \vec{F_2}\right| = \sqrt{\vec{F_1}^2 + \vec{F_2}^2 + \overline{2\vec{F_1} \bullet \vec{F_2}} \cos\alpha}$$

$$= \sqrt{F^2 + F^2 + 2F^2 \cos\alpha}$$

$$= \sqrt{2F^2 + 2F^2 \cos\alpha}$$

$$= \sqrt{2F^2(1 + \cos\alpha)}$$

We know that $\cos^2\phi = \frac{1 + \cos 2\phi}{2}$

Put $\phi = \frac{\alpha}{2}$ *and* $2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$

$$= \sqrt{2F^2 \bullet 2\cos^2\left(\frac{\alpha}{2}\right)}$$

$$= 2F \cos\left(\frac{\alpha}{2}\right)$$

Also $$\phi = \tan^{-1}\left(\frac{F \sin\alpha}{F + F\cos\alpha}\right)$$

$$= \tan^{-1}\left(\frac{F \sin\alpha}{F(1+\cos\alpha)}\right)$$

$$= \tan^{-1}\left(\frac{\sin {}^{\alpha}\!/_{2} \, \cos {}^{\alpha}\!/_{2}}{2 \cos^2 \frac{\alpha}{2}}\right)$$

By $\sin 2A = 2\sin A\cos A$ and put $A = \frac{\alpha}{2}$

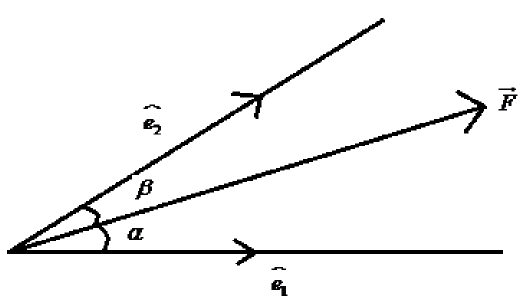
$$\Rightarrow \sin\alpha = 2\sin {}^{\alpha}\!/_{2} \, \cos {}^{\alpha}\!/_{2}$$

$$= \tan^{-1}(\tan {}^{\alpha}\!/_{2})$$

$$\phi = {}^{\alpha}\!/_{2}$$

Problem: 2

To resolve a force $\vec{F}$ into components in two given direction.



Let e_1 and e_2 be unit vectors in two given direction

Let α be the angle between e_1 and $\vec{F}$.

Let β be the angle between $\vec{F}$ and e_2

$\therefore$ $\vec{F}$ can be expressed as

$$\vec{F} = ae_1 + be_2 \tag{1}$$

Multiply e_1 on both sides.

$$e_1 X \vec{F} = a\left(e_1 X e_1\right) + b\left(e_1 X e_2\right)$$

We know that $\vec{a} X \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \sin\phi \ x$

$$\left|e_1\right| X \left|\vec{F}\right| \sin\alpha x = a(0) + b\left|e_1\right|\left|e_2\right| \sin(\alpha+\beta)x \ , \quad \text{since } \vec{i} X \vec{i} = 0$$

$$F \ Sin\alpha \, x = b\sin(\alpha+\beta)x$$

$$\frac{F \ \sin\alpha}{\sin(\alpha+\beta)} = b$$

Multiply e_2 on both sides.

$$e_2 X \vec{F} = a\left(e_2 X e_1\right) + b\left(e_2 X e_2\right)$$

We know that $\vec{a} X \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \sin\phi \ x$

$$\left|e_2\right| X \left|\vec{F}\right| \sin\beta\left(-x\right) = a\left|e_2\right|\left|e_1\right| \sin(\alpha+\beta)\left(-x\right) + b(0)$$

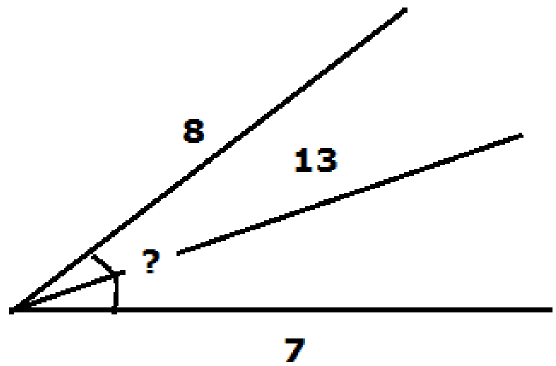
$$F \ Sin\beta(-x) = a\sin(\alpha+\beta)(-x)$$

$$\frac{F \ \sin\beta}{\sin(\alpha+\beta)} = a$$

$\therefore$ Equation (1) becomes

$$\vec{F} = \frac{F \ \sin\beta}{\sin(\alpha+\beta)}\hat{e}_1 + \frac{F \ \sin\alpha}{\sin(\alpha+\beta)}\hat{e}_2$$

Problem 3: If the resultant of two forces meeting at a point with magnitudes 7 and 8 is a force with magnitudes 13, find the angle between the two given forces



Solution:

We know that

The Magnitude of resultant force is $\left|\vec{F_1}+\vec{F_2}\right|^2 = \vec{F_1}^2 + \vec{F_2}^2 + \overline{2\vec{F_1}\bullet\vec{F_2}}\cos\alpha$

The resultant of 7 and 8 is

$$\left|\vec{F_1}+\vec{F_2}\right| = 13$$

$$(13)^2 = (7)^2 + (8)^2 + (2)(7)(8)\cos\alpha$$

$$169 = 113 + 112\ Cos\alpha$$

$$112\cos\alpha = 169 - 113$$

$$112\cos\alpha = 56$$

$$\cos\alpha = \frac{56}{112}$$

$$\cos\alpha = \frac{1}{2}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{1}{2}\right) = 60^{\circ}$$

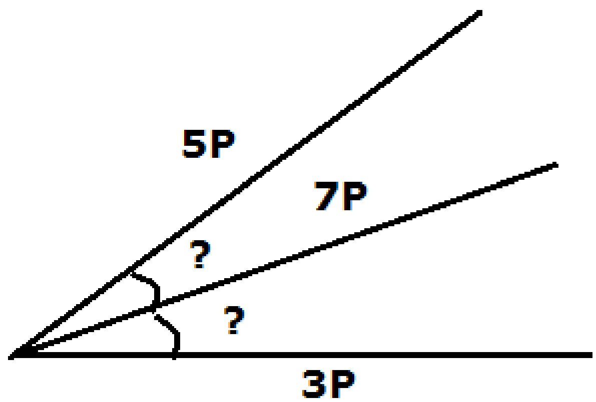
Problem 4: If the resultant of forces 3P, 5P is equal 7P.

Find i) Angle between two forces,

ii) The Angle which the resultant makes with the first force

Solution:

We Know that



Part: 1

The Magnitude off Resultant force is

$$\left|\overrightarrow{F_1} + \overrightarrow{F_2}\right|^2 = \overrightarrow{F_1}^2 + \overrightarrow{F_2}^2 + \overline{2F_1} \bullet \overrightarrow{F_2} \cos\alpha$$

The resultant of 3P and 5P is 7P.

Then

$$\left|\overrightarrow{F_1} + \overrightarrow{F_2}\right| = 7P$$

$$(7\,\mathrm{P})^2 = (3\,\mathrm{P})^2 + (5\,\mathrm{P})^2 + (2)(3\,\mathrm{P})(5\,\mathrm{P})\cos\alpha$$

$$49(\mathrm{P})^2 = 9(\mathrm{P})^2 + 25(\mathrm{P})^2 + (30)(\mathrm{P}^2)\cos\alpha$$

$$Now \div P^2 \qquad \Rightarrow 49 = 34 + 30\cos\alpha$$

$$30\ \cos\alpha = 49 - 34 = 15$$

$$\cos\alpha = \frac{15}{30}$$

$$\cos\alpha = \frac{1}{2}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

Part: 2

The angle between resultant and find force is

$$\phi = \tan^{-1}\left(\frac{F_2 \sin\alpha}{F_1 + F_2 \cos\alpha}\right)$$

$$= \tan^{-1}\left(\frac{5P\sin\alpha}{3P + 5P\cos 60^\circ}\right)$$

$$= \tan^{-1}\left(\frac{5P\sqrt{3/2}}{3P + 5P(1/2)}\right)$$

$$= \tan^{-1}\left(\frac{5P\sqrt{3}/2}{6P + 5P/2}\right)$$

$$= \tan^{-1}\left(\frac{5\cancel{P}\sqrt{3}}{11\cancel{P}}\right) \text{ Cancel P}$$

$$\phi = \tan^{-1}\left(\frac{5\sqrt{3}}{11}\right)$$

Problem 5:

The Magnitude of the resultant of two given Forces P, Q, is R. If Q is Doubled, then R is doubled. If Q is reversed then also R is Doubled, Show that $P:Q:R=\sqrt{2}:\sqrt{3}:\sqrt{2}$

Proof:

We Know that, The magnitude of Resultant force is

$$\left|\overrightarrow{F_1}+\overrightarrow{F_2}\right|^2=\overrightarrow{F_1}^2+\overrightarrow{F_2}^2+\overline{2\overrightarrow{F_1}\bullet\overrightarrow{F_2}}\cos\alpha$$

Given the resultant of P,Q is R

$$\therefore \left|\overrightarrow{F_1}+\overrightarrow{F_2}\right|=R$$
$$\overrightarrow{F_1}=P$$
$$\overrightarrow{F_2}=Q$$

$$R^2=P^2+Q^2+2PQ\cos\alpha \qquad (1)$$

If Q is Doubled, R is Doubled

$$\therefore \left|\overrightarrow{F_1}+\overrightarrow{F_2}\right|=2R$$
$$\overrightarrow{F_1}=P$$
$$\overrightarrow{F_2}=2Q$$

$$(2R)^2=P^2+(2Q)^2+2P(2Q)\cos\alpha$$

$$(4R^2)=P^2+4Q^2+4PQ\cos\alpha \qquad (2)$$

If Q is reversed, Then also R is doubled

$$\therefore \left|\overrightarrow{F_1}+\overrightarrow{F_2}\right|=2R$$
$$\overrightarrow{F_1}=P$$
$$\overrightarrow{F_2}=-Q$$

$$(2R)^2=P^2+(-Q)^2+2P(-Q)\cos\alpha$$

$$(4R^2) = P^2 + Q^2 - 2PQ\cos\alpha \qquad (3)$$

(1)+(3)=>

$$P^2 + Q^2 + 2PQ\cos\alpha = (R^2)$$
$$P^2 + Q^2 - 2PQ\cos\alpha = (4R^2)$$
$$----------------$$
$$2P^2 + 2Q^2 \qquad = (5R^2) ------------------(4)$$

(2)+(2X(3))=>

$$2P^2 + 2Q^2 - 4PQ\cos\alpha = (8R^2)$$
$$P^2 \quad + Q^2 \quad - 2PQ\cos\alpha = (4R^2)$$
$$----------------$$
$$3P^2 + 6Q^2 \qquad = (12R^2)$$
$$\div 3$$
$$P^2 + 2Q^2 \qquad = (4R^2) -------------------(5)$$

Solve (4) and (5)

$$(4) - (5) \Rightarrow 2P^2 + 2Q^2 = 5R^2$$
$$P^2 + 2Q^2 = 4R^2$$
$$------------$$
$$P^2 \qquad = R^2$$

Put $P^2 = R^2$ in (4)

$$2(\mathrm{R}^2) + 2\mathrm{Q}^2 = 5R^2$$
$$2\mathrm{Q}^2 = 5R^2 - 2(\mathrm{R}^2)$$
$$2\mathrm{Q}^2 = 3R^2$$
$$\mathrm{Q}^2 = \frac{3}{2}R^2$$

$$\therefore P^2 : Q^2 : R^2 = R^2 : \frac{3}{2}R^2 : R^2$$
$$= 2R^2 : 3R^2 : 2R^2$$
$$P^2 : Q^2 : R^2 = 2 : 3 : 2$$
$$P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$$

Hence the Proof

Problem 6: The Resultant of two force P,Q, is R, If P is doubled, then R is doubled. If Q is doubled and reversed, then also R is doubled. Show that $P:Q:R=\sqrt{6}:\sqrt{2}:\sqrt{5}$

Proof:

We Know that , the magnitude of Resultant force is

$$\left|\overrightarrow{F_1}+\overrightarrow{F_2}\right|^2 = \overrightarrow{F_1}^2+\overrightarrow{F_2}^2+\overline{2\overrightarrow{F_1}\bullet\overrightarrow{F_2}}\cos\alpha$$

Given the resultant of P,Q is R

$$\therefore \left|\overrightarrow{F_1}+\overrightarrow{F_2}\right| = R$$
$$\overrightarrow{F_1}=P$$
$$\overrightarrow{F_2}=Q$$

$$R^2 = P^2+Q^2+2PQ\cos\alpha \qquad (1)$$

If P is Doubled, R is Doubled

$$\therefore \left|\overrightarrow{F_1}+\overrightarrow{F_2}\right| = 2R$$
$$\overrightarrow{F_1}=2P$$
$$\overrightarrow{F_2}=Q$$

$$(2R)^2=(2P^2)+(Q)^2+2(2P)(Q)\cos\alpha$$

$$(4R^2)=4P^2+Q^2+4PQ\cos\alpha \qquad (2)$$

If Q is doubled, Then also R is also doubled

$$\therefore \left|\overrightarrow{F_1}+\overrightarrow{F_2}\right| = 2R$$
$$\overrightarrow{F_1}=P$$
$$\overrightarrow{F_2}=2Q$$

$$(4R)^2 = P^2+(4Q)^2+4PQ\cos\alpha \qquad (3)$$

If Q is Reversed, Then also R is Doubled then

$$\therefore \left|\overrightarrow{F_1}+\overrightarrow{F_2}\right| = 2R$$

$$\overrightarrow{F_1} = P$$

$$\overrightarrow{F_2} = -Q$$

$$(2R)^2 = P^2 + (-Q)^2 + 2P(-Q)\cos\alpha$$

$$(4R^2) = P^2 + Q^2 - 2PQ\cos\alpha \qquad (4)$$

Solve Equation (1), (2) and (3) then we get

$$P : Q : R = \sqrt{6} : \sqrt{2} : \sqrt{5}$$

Problem 7: Two equal forces are implined at an angles magnitude of there resultant is 3 times the magnitude of resultant when the forces are inclined at an angle 2ϕ. S.T $\cos\theta = 3\cos\phi$

Proof:

We Know that, the magnitude of the Resultant force is

$$\left|\overrightarrow{F_1}+\overrightarrow{F_2}\right| = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos 2}$$

Given the Magnitude of two equal forces at angle 2ϑ= 3{times Magnitude at angle)

$$\therefore \sqrt{P^2 + P^2 + 2P^2\cos\theta} = 3\sqrt{P^2 + P^2 + 2P^2\cos\phi}$$

$$\sqrt{2P^2 + 2P^2\cos 2\theta} = 3\sqrt{2P^2 + 2P^2\cos 2\phi}$$

$$\sqrt{2P^2(1+\cos 2\theta)} = 3\sqrt{2P^2(1+\cos 2\phi)}$$

$$\sqrt{2P^2(2\cos^2\theta)} = 3\sqrt{2P^2(2\cos^2\phi)}$$

$$2P\cos\theta = 3(2\,\mathrm{P}\cos\phi)$$

$$\cos\theta = 3\cos\phi$$

Problem 8: The Resultant of two Forces of Magnitudes P and Q acting at a point as Magnitudes $(2n+1)\sqrt{P^2+Q^2}$ and $(2n-1)\sqrt{P^2+Q^2}$ When the forces are inclined at α and $90^0-\alpha$ respectively. S.T $\tan\alpha=\dfrac{n-1}{n+1}$

Proof:

We Know that, the magnitude of the Resultant force is

$$\left|\overrightarrow{F_1}+\overrightarrow{F_2}\right|=\sqrt{F_1^2+F_2^2+2F_1F_2\cos 2}$$

Since the Resultant of Magnitude of P and Q is $(2n+1)\sqrt{P^2+Q^2}$ at an angle α

$$\sqrt{P^2+Q^2+2PQ\cos\alpha}=(2n+1)\sqrt{P^2+Q^2}$$

Solving on both sides

$$(P^2+Q^2)+2PQ\cos\alpha=(2n+1)^2\sqrt{P^2+Q^2}$$

$$2PQ\cos\alpha=(4n^2+4n+1)(P^2+Q^2)-(P^2+Q^2)$$

$$=(4n^2+4n+1-1)(P^2+Q^2)$$

$$2PQ\cos\alpha=4n(n+1)(P^2+Q^2)---- \rightarrow (1)$$

Also given the Magnitude of P and Q is $(2n-1)\sqrt{P^2+Q^2}$ at an angle $90^\circ-\alpha$

$$\sqrt{P^2+Q^2+2PQ\cos(90-\alpha)}=(2n-1)\sqrt{P^2+Q^2}$$

$$\sqrt{P^2+Q^2+2PQ\sin\alpha}=(2n-1)\sqrt{P^2+Q^2}$$

Squaring, we get

$$P^2+Q^2+2PQ\sin\alpha=(2n-1)^2(P^2+Q^2)$$

$$2PQ\sin\alpha=(4n^2-4n-1)(P^2+Q^2)-(P^2+Q^2)$$

$$2\,PQ\sin\alpha = (4n^2 - 4n - 1 + 1)(P^2 + Q^2)$$

$$2\,PQ\sin\alpha = 4n(n-1)(P^2 + Q^2) ----\to (2)$$

$(2) \div (1)$

$$\frac{2\,PQ\sin\alpha = 4n(n-1)(P^2 + Q^2)}{2\,PQ\cos\alpha = 4n(n+1)(P^2 + Q^2)}$$

$$\tan\alpha = \frac{n-1}{n+1}$$

Problem 9: The Magnitude of resultant of the forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ acting on a particle is equal to the Magnitude $\overrightarrow{F_1}$, when the first force is doubled, S.T the new resultant is perpendicular to $\overrightarrow{F_2}$.

Proof:

Given,

The Magnitude of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ = Magnitude of $\overrightarrow{F_1}$

$$\therefore \left|\overrightarrow{F_1} + \overrightarrow{F_2}\right| = \left|\overrightarrow{F_1}\right|$$

$$= \left|\overrightarrow{F_1} + \overrightarrow{F_2}\right|^2 = \left|\overrightarrow{F_1}\right|^2$$

$$(\overrightarrow{F_1} + \overrightarrow{F_2}).(\overrightarrow{F_1} + \overrightarrow{F_2}) = (\overrightarrow{F_1}.\overrightarrow{F_1})$$

$$\overrightarrow{F_1}.\overrightarrow{F_1} + \overrightarrow{F_1}.\overrightarrow{F_2} + \overrightarrow{F_2}.\overrightarrow{F_1} + (\overrightarrow{F_2}.\overrightarrow{F_2}) = \overrightarrow{F_1}.\overrightarrow{F_1}$$

$$\overrightarrow{F_1}.\overrightarrow{F_2} + \overrightarrow{F_1}.\overrightarrow{F_2} + (\overrightarrow{F_2}.\overrightarrow{F_2}) = 0$$

$$2\overrightarrow{F_1}.\overrightarrow{F_2} + (\overrightarrow{F_2}.\overrightarrow{F_2}) = 0$$

$$(2\overrightarrow{F_1} + \overrightarrow{F_2}).\overrightarrow{F_2} = 0$$

$\therefore 2\overrightarrow{F_1} + \overrightarrow{F_2}$ is Perpendicular to $\overrightarrow{F_2}$

$\therefore$ The Resultant is Perpendicular to $\overrightarrow{F_2}$

Problem 10: The resultant of two forces P,Q is of Magnitude P. Such that if P is double, the new resultant is perpendicular to the force Q and its Magnitude is $\sqrt{4P^2 - Q^2}$

Solution:

Given the resultant of P,Q=magnitude of P

$$
\begin{aligned}
ie) \quad |P+Q| &= |P| \\
\Rightarrow \quad |P+Q|^2 &= |P|^2 \\
(P+Q)(P+Q) &= P^2 \\
\cancel{P^2} + PQ + QP + Q^2 &= \cancel{P^2}
\end{aligned}
$$

$$P.Q + P.Q + Q.Q = 0$$

$$2P.Q + Q.Q = 0 \tag{1}$$

$$(2P + Q).Q = 0$$

The resultant of 2P and Q is $\perp r$ to Q

The Magnitude of new resultant is (The resultant of 2P and Q)=$|2P+Q|$

$$
\begin{aligned}
\text{ie)} \quad |2P+Q|^2 &= (2P+Q).(2\mathrm{P}+\mathrm{Q}) \\
&= (2\mathrm{P}+\mathrm{Q}).2\mathrm{P} + (2\mathrm{P}+\mathrm{Q}).\mathrm{Q} \\
&= (2\mathrm{P}.2\mathrm{P}) + \mathrm{Q}.2\mathrm{P} \\
&= 4\mathrm{P}^2 + 2P.Q \\
&= 4P^2 - Q.Q \text{ (From (1))}
\end{aligned}
$$

Since

$$2P.Q + Q.Q = 0$$

$$2P.Q = -Q.Q$$

$$= 4P^2 - Q^2$$

$$\left|2P + Q\right| = \sqrt{4P^2 - Q^2}$$

Problem 11: If two Force $\vec{P}, \vec{Q}$ acting at a force is such that there sum and difference are perpendicular to each other. S.T P=Q

Proof:

The Sum and difference of two forces

$$\vec{P}, \vec{Q} \text{ is } \vec{P} + \vec{Q} \text{ and } \vec{P} - \vec{Q}$$

Since sum and difference are perpendicular to each other

$$\therefore \qquad (\vec{P} + \vec{Q})(\vec{P} - \vec{Q}) = 0$$

$$\vec{P}.\vec{P} - \vec{P}.\vec{Q} + \vec{Q}.\vec{P} - \vec{Q}.\vec{Q} = 0$$

$$P^2 - \cancel{\vec{P}.\vec{Q}} + \cancel{\vec{P}.\vec{Q}} - Q^2 = 0$$

$$P^2 - Q^2 = 0$$

$$P^2 = Q^2$$

$$P = Q$$

WEIGHT: OF A BODY

Definition: The force of attraction of the earth on a body is called Weight.

TENSION OF A FORCE:

Definition: Tension is the State of beings Stretched tight.

HOORE'S LAW

Tension of an elastic (body) string varies as the ratio of the extension of string beyond its natural Length.

$$\text{ie. Tension} = \lambda \frac{Extension}{NaturalLength}$$

Where λ is called coefficient of elasticity.

LIMITING FRICTION

Definition: The maximum Friction that can be generated between true static surfaces in contact each other, is called Limiting Friction. It is denoted by F.

EQULIBRIUM OF A PARTICLE FORCE:

The resultant of the forces acting at a point is 0(Zero) then the Force are said to be Equlibrium.

NOTE:

When three Forces acting on a triangle ABC

$$i)\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{CA}} = 0$$

$ii)$ If M is the Midpoint of $\overrightarrow{\mathrm{BC}}$

Then

$$\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{AC}})$$

$$=$$

$$\frac{1}{2}(\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{CA}})$$

$$\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB},$$

$$\hat{BC}, \hat{CA}, \hat{AB}$$

$$\vec{F} = F_1\, \hat{BC} + F_2\, \hat{CA} + F_3\, \hat{AB}$$

$$\sqrt{F_1^2 + F_2^2 + F_3^2 - 2F_2F_3 \cos A - 2F_3F_1 \cos B - 2F_1F_2 \cos C}$$

$$\hat{AB}.\hat{AB} = \hat{BC}.\hat{BC} = \hat{CA}.\hat{CA} = 1$$

$$i.i = 1$$

$$\left|\vec{F}\right|^2 = \vec{F}.\vec{F}$$

$$= (F_1\, \hat{BC} + F_2\, \hat{CA} + F_3\, \hat{AB}).(F_1\, \hat{BC} + F_2\, \hat{CA} + F_3\, \hat{AB})$$

$$= \mathrm{F}_1^2 + F_1F_2\, \hat{BC}.\hat{CA}. + F_1F_3\, \hat{BC}.\hat{AB} + F_1F_2\, \hat{CA}.\hat{BC}$$

$$+ \mathrm{F}_2^2 + F_2F_3\, \hat{CA}.\hat{AB}. + F_1F_3\, \hat{AB}.\hat{BC} + F_2F_3\, \hat{AB}.\hat{CA} + \mathrm{F}_3^2$$

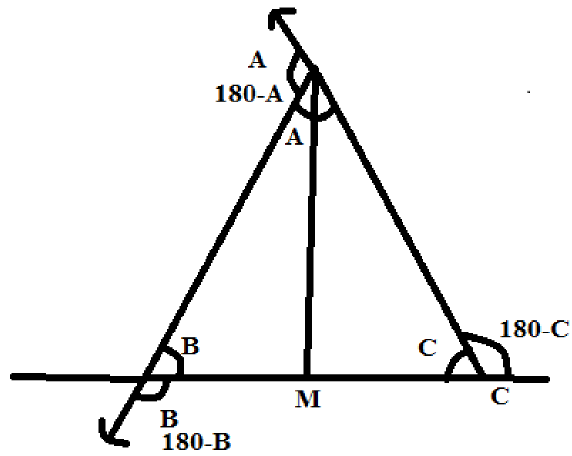
$$= \mathrm{F}_1^2 + \mathrm{F}_2^2 + \mathrm{F}_3^2 + F_1F_2 \cos(180^0 - C) + F_1F_2 \cos(180^0 - B) + F_1F_2 \cos(180^0 - C)$$

$$+ F_2F_3 \cos(180^0 - A) + F_1F_3 \cos(180^0 - B) + F_2F_3 \cos(180^0 - A)$$

$$= \mathrm{F}_1^2 + \mathrm{F}_2^2 + \mathrm{F}_3^2 - F_1F_2 \cos C - F_1F_3 \cos\mathrm{B} - F_1F_2 \cos C - F_2F_3 \cos\mathrm{A} - F_1F_3 \cos\mathrm{B} - F_2F_3 \cos\mathrm{A}$$

$$= \mathrm{F}_1^2 + \mathrm{F}_2^2 + \mathrm{F}_3^2 - 2F_2F_3 \cos\mathrm{A} - 2F_1F_3 \cos\mathrm{B} - 2F_1F_2 \cos C$$

$$\left|\vec{F}\right| = \sqrt{F_1^2 + F_2^2 + F_3^2 - 2F_2F_3 \cos A - 2F_3F_1 \cos B - 2F_1F_2 \cos C}$$

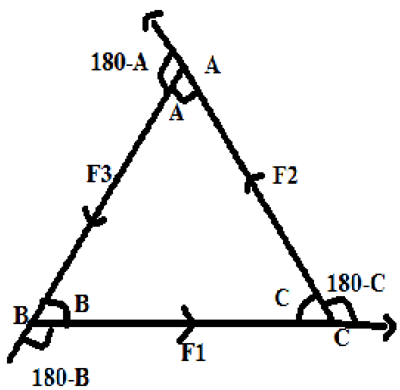


Problem:

Forces of Magnitude $F_1, F_{2,} F_3$ act on a Particle. If their direction are parallel to $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB},$

Where ABC is a triangle show that the Magnitude of their resultant is

$$\sqrt{F_1^2 + F_2^2 + F_3^2 - 2F_2F_3 \cos A - 2F_3F_1 \cos B - 2F_1F_2 \cos C}$$



Let $\hat{BC}, \hat{CA}, \hat{AB}$ are the unit vector along $\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB}$

Let F be the Magnitude of the resultant. Then

$$\vec{F} = F_1 \hat{BC} + F_2 \hat{CA} + F_3 \hat{AB}$$

The Magnitude of the Resultant is

$$\hat{AB}.\hat{AB}=\hat{BC}.\hat{BC}=\hat{CA}.\hat{CA}=1$$

$i.i=1$

$$\left|\vec{F}\right|^2=\vec{F}.\vec{F}$$

$$=(F_1\,\hat{BC}+F_2\,\hat{CA}+F_3\,\hat{AB}).(F_1\,\hat{BC}+F_2\,\hat{CA}+F_3\,\hat{AB})$$

$$=\mathrm{F}_1^2+F_1F_2\,\hat{BC}.\hat{CA}.+F_1F_3\,\hat{BC}.\hat{AB}+F_1F_2\,\hat{CA}.\hat{BC}$$

$$+\mathrm{F}_2^2+F_2F_3\,\hat{CA}.\hat{AB}.+F_1F_3\,\hat{AB}.\hat{BC}+F_2F_3\,\hat{AB}.\hat{CA}+\mathrm{F}_3^2$$

$$=\mathrm{F}_1^2+\mathrm{F}_2^2+\mathrm{F}_3^2+F_1F_2\cos(180^0-C)+F_1F_2\cos(180^0-B)+F_1F_2\cos(180^0-C)$$

$$+F_2F_3\cos(180^0-A)+F_1F_3\cos(180^0-B)+F_2F_3\cos(180^0-A)$$

$$=\mathrm{F}_1^2+\mathrm{F}_2^2+\mathrm{F}_3^2-F_1F_2\cos C-F_1F_3\cos\mathrm{B}-F_1F_2\cos C-F_2F_3\cos\mathrm{A}-F_1F_3\cos\mathrm{B}-F_2F_3\cos\mathrm{A}$$

$$=\mathrm{F}_1^2+\mathrm{F}_2^2+\mathrm{F}_3^2-2F_2F_3\cos\mathrm{A}-2F_1F_3\cos\mathrm{B}-2F_1F_2\cos C$$

$$\left|\vec{F}\right|=\sqrt{F_1^2+F_2^2+F_3^2-2F_2F_3\cos A-2F_3F_1\cos B-2F_1F_2\cos C}$$

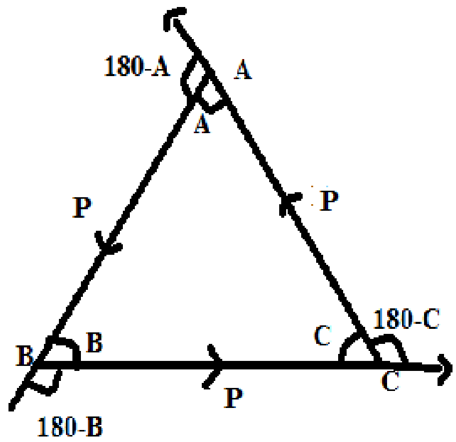
Problem: There Forces of equal Magnitudes P act on a Particle. If their directions are parallel to the sides BC, CA, AB of a triangle ABC. S.T the Magnitude of their resultant

$$P\sqrt{3-2\cos A-2\cos B-2\cos\mathrm{C}}$$

Solution

Since three forces have equal magnitude P.

Let BC, CA, AB be the unit vectors along $\overrightarrow{\mathrm{BC}}$, $\overrightarrow{\mathrm{CA}}$, $\overrightarrow{\mathrm{AB}}$



$$\vec{F} = P\,\overrightarrow{BC} + P\,\overrightarrow{CA} + P\,\overrightarrow{AB}$$

The magnitude of their resultant is

$$\left|\vec{F}\right| = \left(P\,\overrightarrow{BC} + P\,\overrightarrow{CA} + P\,\overrightarrow{AB}\right)\cdot\left(P\,\overrightarrow{BC} + P\,\overrightarrow{CA} + P\,\overrightarrow{AB}\right)$$

$$\begin{aligned} &= 3P^2 + P^2\cos(180-\mathrm{C}) + P^2\cos(180-\mathrm{B}) + P^2\cos(180-C) + \\ &\quad P^2\cos(180-\mathrm{A}) + P^2\cos(180-\mathrm{B}) + P^2\cos(180-\mathrm{A}) \end{aligned}$$

$$= 3P^2 - P^2\cos(\mathrm{C}) - P^2\cos(\mathrm{B}) - P^2\cos(C) + P^2\cos(\mathrm{A}) + P^2\cos(\mathrm{B}) + P^2\cos(\mathrm{A})$$

$$\left|\vec{F}\right| = 3P^2 - 2P^2\cos(\mathrm{A}) - 2P^2\cos(\mathrm{B}) - 2P^2\cos(C)$$

$$\begin{aligned} \vec{F} &= \sqrt{P^2\left(3 - 2\cos(\mathrm{A}) - 2\cos(\mathrm{B}) - 2\cos(C)\right)} \\ &= P\sqrt{\left(3 - 2\cos(\mathrm{A}) - 2\cos(\mathrm{B}) - 2\cos(C)\right)} \end{aligned} \quad \text{(OR)}$$

We know that The magnitude of resultant of three forces is

$$\vec{F} = \sqrt{F_1^2 + F_2^2 + F_3^2 - 2F_3F_2\cos A - 2F_1F_3\cos\mathrm{B} - 2F_1F_2\cos\mathrm{C}}$$

Since the magnitude of the three forces is P

$$ie)\ F_1 = P, F_2 = P, F_3 = P,$$

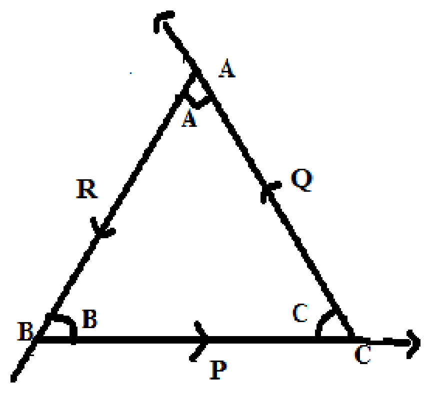
$$\vec{F} = \sqrt{P^2 + P^2 + P^2 - 2PP\cos A - 2PP\cos\mathrm{B} - 2PP\cos\mathrm{C}}$$

$$\vec{F} = \sqrt{P^2(3-2\cos A-2\cos \mathrm{B}-2\cos \mathrm{C})}$$

$$\vec{F} = P\sqrt{(3-2\cos(\mathrm{A})-2\cos(\mathrm{B})-2\cos(C))}$$

Problem: Three forces acting at a point are parallel to the sides of a triangle ABC, taken in order and in magnitude there are proportional to the cosines of the opposite angles such that the magnitude of the resultant is proportional to $\sqrt{1-8\cos A\cos B\cos C}$

Solution:



Let P,Q,R be the three forces acting parallel to the sides of a triangle ABC

Given magnitude are proportional to the cosines.

$$\begin{aligned} ie)\ P\ \alpha\ \cos A &\Rightarrow P = K\cos A \\ Q\ \alpha\ \cos \mathrm{B} &\Rightarrow Q = K\cos \mathrm{B} \\ R\ \alpha\ \cos\ \mathrm{C} &\Rightarrow R = K\cos\ \mathrm{C} \end{aligned}$$

We know that , the magnitude of resultant of three forces is

$$\vec{F} = \sqrt{F_1^2+F_2^2+F_3^2-2F_3F_2\cos A-2F_1F_3\cos \mathrm{B}-2F_1F_2\cos \mathrm{C}}$$

Since the magnitude of the three forces is P

$ie)\ F_1 = P, F_2 = Q, F_3 = R,$

$$\vec{F} = \sqrt{P^2+Q^2+R^2-2QR\cos A-2PR\cos \mathrm{B}-2PQ\cos \mathrm{C}}$$

$$\vec{F} = \sqrt{\begin{array}{l} K^2\cos^2 A + K^2\cos^2 B + K^2\cos^2 C - 2K^2\cos B\cos C\cos A \\ -2K^2\cos\mathrm{A}\cos C\cos\mathrm{B} - 2K^2\cos A\cos B\cos\mathrm{C} \end{array}}$$

$$\vec{F} = \sqrt{K^2\cos^2 A + K^2\cos^2 B + K^2\cos^2 C - 6K^2\cos B\cos C\cos A}$$

$$\vec{F} = \sqrt{K^2(\cos^2 A + \cos^2 B + \cos^2 C - 6\cos B\cos C\cos A)}$$

$$\vec{F} = K\sqrt{(\cos^2 A + \cos^2 B + \cos^2 C - 6\cos B\cos C\cos A)}$$

$$\vec{F} = K\sqrt{1 - 2\cos B\cos C\cos A - 6\cos B\cos C\cos A)}$$

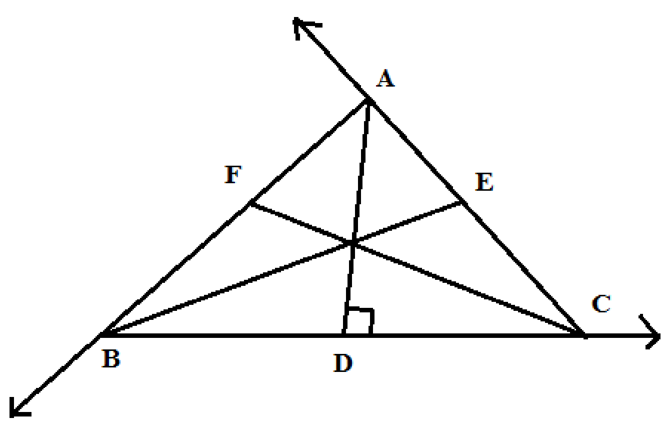
$$\vec{F} = K\sqrt{1 - 8\cos B\cos C\cos A} \text{ (OR)}$$

$$\vec{F} \;\alpha \sqrt{1 - 8\cos B\cos C\cos A}$$

Problem:

The Sides BC, CA, AB of a triangle ABC are bisected in D,E,F, Such that the forces represented by DA, EB, FC are in equilibrium.

Solution:



Let the forces through the centroid. Since D is the midpoint of BC

$$\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$$

$$-\overrightarrow{DA} = \frac{1}{2}(\overrightarrow{AB} - \overrightarrow{CA}) \text{ since } \overrightarrow{\mathrm{AD}} = -\overrightarrow{DA}$$

$$\overrightarrow{BE} = \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{BA})$$

$$= \frac{1}{2}(\overrightarrow{BC} - \overrightarrow{AB}) \text{ since } \overrightarrow{EB} = -\overrightarrow{BE}$$

$$\overrightarrow{EB} = -\frac{1}{2}(\overrightarrow{BC} - \overrightarrow{AB})$$

Also F is the midpoint of AB

$$\overrightarrow{CF} = \frac{1}{2}(\overrightarrow{CA} + \overrightarrow{CB})$$

$$= \frac{1}{2}(\overrightarrow{CA} - \overrightarrow{BC}) \text{ since } \overrightarrow{CA} = -\overrightarrow{AC}$$

$$\overrightarrow{FC} = -\frac{1}{2}(\overrightarrow{CA} - \overrightarrow{BC})$$

$$\overrightarrow{DA} + \overrightarrow{EB} + \overrightarrow{FC} = -\frac{1}{2}(\overrightarrow{AB} - \overrightarrow{CA} + \overrightarrow{BC} - \overrightarrow{AB} + \overrightarrow{CA} - \overrightarrow{BC})$$

$$= 0$$

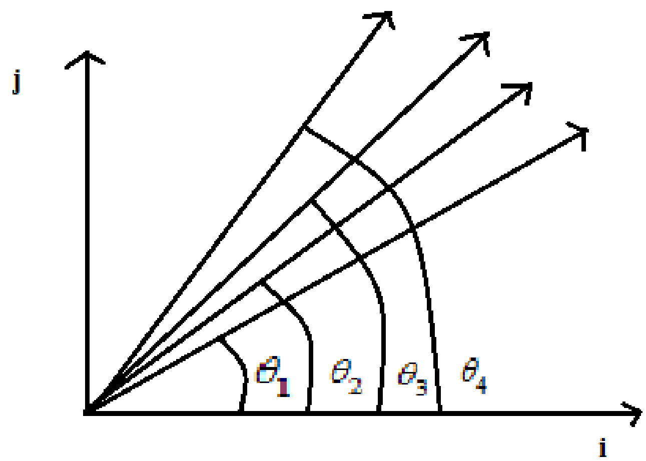
The force DA, EB, FC are in equilibrium resultant of several forces acting at a particular.

RESULTANT OF SEVERAL FORCES ACTING ON A PLAN

Problem:

Fine the resultant of coplanar forces using their components

Solution



Let $\vec{i}$ and $\vec{j}$ be the unit vectors

Let $\overrightarrow{F_1}, \overrightarrow{F_2}, \overrightarrow{F_3}, \overrightarrow{F_4} ... \overrightarrow{F_n}$ be the several forces acting at a point

$\therefore$ the resultant is $\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} + ... + \overrightarrow{F_n}$

Taking dot product with $\vec{i}$

$$\vec{R} \bullet \vec{i} = \overrightarrow{F_1} \bullet \vec{i} + \overrightarrow{F_2} \bullet \vec{i} + \overrightarrow{F_3} \bullet \vec{i} + ... + \overrightarrow{F_n} \bullet \vec{i}$$

$$R \cos\theta = F_1 \cos\theta_1 + F_2 \cos\theta_2 + ...F_n \cos\theta_n$$

$$R \cos\theta = X \ \ (Say) \text{ --------------------------(1)}$$

where $X = F_1 \cos\theta_1 + F_2 \cos\theta_2 + ...F_n \cos\theta_n$

also Taking dot product with $\vec{j}$

$$\vec{R} \bullet \vec{j} = \overrightarrow{F_1} \bullet \vec{j} + \overrightarrow{F_2} \bullet \vec{j} + \overrightarrow{F_3} \bullet \vec{j} + ... + \overrightarrow{F_n} \bullet \vec{j}$$

$$R \sin\theta = F_1 \sin\theta_1 + F_2 \sin\theta_2 + ...F_n \cos\theta_n$$

$$R \sin\theta = Y \ (say) \text{------------------------------------(2)}$$

Where $Y = F_1 \sin\theta_1 + F_2 \sin\theta_2 + ...F_n \cos\theta_n$

$(1)^2 + (2)^2$

$$R^2\cos^2\theta + R^2\sin^2\theta = X^2 + Y^2$$
$$R^2\left(\cos^2\theta + \sin^2\theta\right) = X^2 + Y^2$$
$$R^2 = X^2 + Y^2$$
$$R = \sqrt{X^2 + Y^2}$$

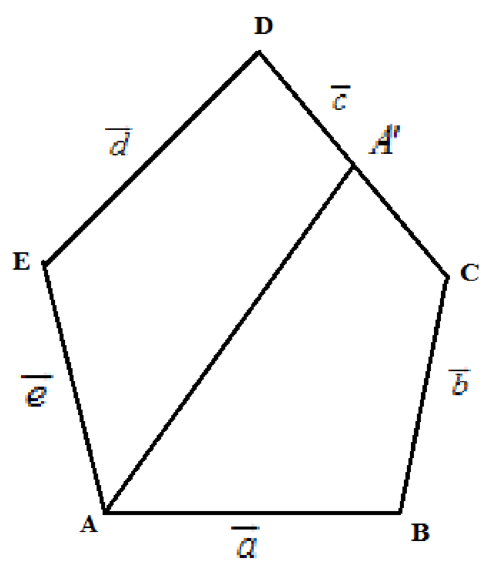
Now angle between forces

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Problem :

Five forces acting at a point are represented in magnitude and direction by the line joining the vertices of any pentagon to the midpoint of their opposite side. Show that there are in equilibrium.

Solution:



Let $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EF}$ be the sides of the pentagon and A', B', C', D', E' be the midpoints of the opposite sides.

Let $\overrightarrow{AB}=\vec{\mathrm{a}}, \overrightarrow{BC}=\vec{\mathrm{b}}, \overrightarrow{CD}=\vec{\mathrm{c}}, \overrightarrow{DE}=\vec{\mathrm{d}}, \overrightarrow{EF}=\vec{e}$

Since A' is the midpoint of opposite side of A

$$\overrightarrow{AA'}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA'}=\vec{\mathrm{a}}+\vec{\mathrm{b}}+\frac{1}{2}\vec{\mathrm{c}}$$

$$\overrightarrow{BB'}=\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DB'}=\vec{\mathrm{b}}+\vec{\mathrm{c}}+\frac{1}{2}\vec{\mathrm{d}}$$

$$\overrightarrow{CC'}=\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EC'}=\vec{\mathrm{c}}+\vec{\mathrm{d}}+\frac{1}{2}\vec{e}$$

$$\overrightarrow{DD'}=\overrightarrow{DE}+\overrightarrow{EA}+\overrightarrow{AD'}=\vec{\mathrm{d}}+\vec{e}+\frac{1}{2}\vec{\mathrm{a}}$$

$$\overrightarrow{EE'}=\overrightarrow{EA}+\overrightarrow{AB}+\overrightarrow{BE'}=\vec{e}+\vec{\mathrm{a}}+\frac{1}{2}\vec{\mathrm{b}}$$

Now,

$$\overrightarrow{AA'}+\overrightarrow{BB'}+\overrightarrow{CC'}+\overrightarrow{DD'}+\overrightarrow{EE'}=\vec{\mathrm{a}}+\vec{\mathrm{b}}+\frac{1}{2}\vec{\mathrm{c}}+\vec{\mathrm{b}}+\vec{\mathrm{c}}+\frac{1}{2}\vec{\mathrm{d}}+\vec{\mathrm{c}}+\vec{\mathrm{d}}+\frac{1}{2}\vec{e}+\vec{\mathrm{d}}+\vec{e}+\frac{1}{2}\vec{\mathrm{a}}+\vec{e}+\vec{\mathrm{a}}+\frac{1}{2}\vec{\mathrm{b}}$$

$$=2\vec{\mathrm{a}}+\frac{\vec{a}}{2}+2\vec{\mathrm{b}}+\frac{\vec{\mathrm{b}}}{2}+2\vec{\mathrm{c}}+\frac{\vec{c}}{2}+2\vec{\mathrm{d}}+\frac{\vec{\mathrm{d}}}{2}+2\vec{e}+\frac{\vec{e}}{2}$$

$$=\frac{5\vec{a}}{2}+\frac{5\vec{\mathrm{b}}}{2}+\frac{5\vec{c}}{2}+\frac{5\vec{\mathrm{d}}}{2}+\frac{5\vec{e}}{2}$$

$$=\frac{5}{2}\left(\vec{a}+\vec{\mathrm{b}}+\vec{c}+\vec{\mathrm{d}}+\vec{e}\right)$$

$$=\frac{5}{2}(0)\ \sin ce\ \vec{a}+\vec{\mathrm{b}}+\vec{c}+\vec{\mathrm{d}}+\vec{e}=0$$

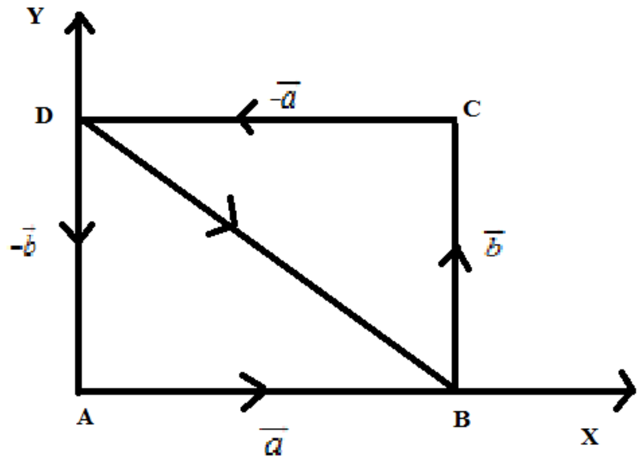
$$\therefore \overrightarrow{AA'}+\overrightarrow{BB'}+\overrightarrow{CC'}+\overrightarrow{DD'}+\overrightarrow{EE'}=0$$

$\therefore$ The five forces are in equilibrium.

Problem:

The Forces ending at point represented magnitude and direction by $\overrightarrow{AB}, 2\overrightarrow{BC}, 2\overrightarrow{CD}, \overrightarrow{DA}, \overrightarrow{DB}$, where ABCD is a square such that the forces are in equilibrium.

Solution:



Let $\overrightarrow{AB}=\vec{a},\ \overrightarrow{BC}=\vec{b}$

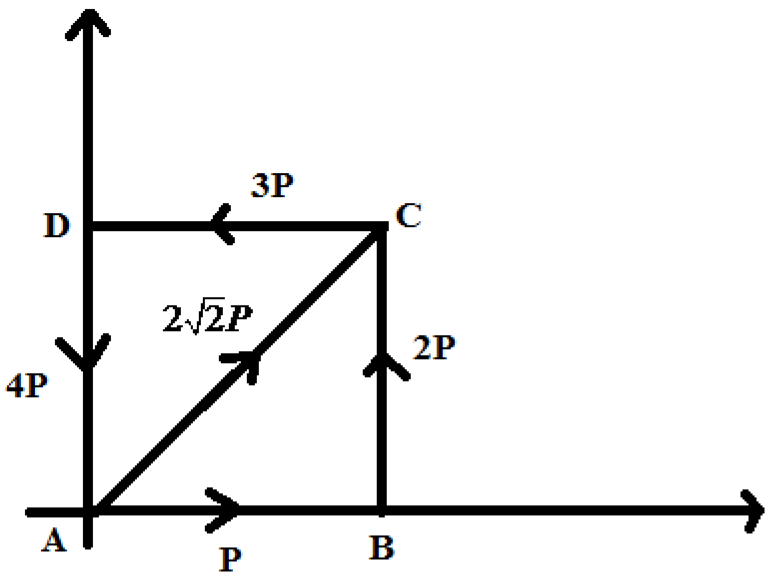
Now $\overrightarrow{AB}+2\overrightarrow{BC}+2\overrightarrow{CD}+\overrightarrow{DA}+\overrightarrow{DB}=\vec{a}+2\vec{b}+2(-\vec{a})+(-\vec{b})+(-\vec{b}+\vec{a})$

$$=\vec{a}+2\vec{b}-2\vec{a}-\vec{b}-\vec{b}+\vec{a}$$
$$=0$$

Therefore the forces are in equilibrium.

Problem: Let P,2P,3P,4P,$2\sqrt{2}$P acting along the sides AB,BC,CD,DA and AC of the square A,B,C,D acting at a point. Such that the forces are in equilibrium.

Solution:



Let P,2P,3P,4P,$2\sqrt{2}$P be the forces acting along the sides AB,BC,CD,DA and AC

The forces acting horizontally

$$\begin{aligned} X &= P\cos 0^\circ + 3P\cos 180^\circ + 2\sqrt{2}P\cos 45^\circ \\ &= P - 3P + 2\cancel{\sqrt{2}}P(1/\cancel{\sqrt{2}}) \\ &= -2\mathrm{P} + 2P \\ &= 0 \end{aligned}$$

Similarly the forces acting along vertically

$$\begin{aligned} Y &= 2P - 4P + 2\sqrt{2}P\sin 45^\circ \\ &= 2P - 4P + 2\cancel{\sqrt{2}}P(1/\cancel{\sqrt{2}}) \\ &= -2\mathrm{P} + 2P \\ &= 0 \end{aligned}$$

Therefore the forces are in equilibrium

The resultant is

$$R = \sqrt{X^2 + Y^2} = \sqrt{0} = 0$$

EQUILIBRIUM OF A PARTICLE UNDER THREE FORCES.

Triangle law of forces:

If three forces acting on a particle can be represented in magnitude and direction by the sides of a triangle taken in order then the forces keep the particle in equilibrium.

Polygon law of forces:

If several forces acting on a particle can be represented in magnitude and direction by the sides of a polygon taken in order, then the forces keep the particle in equilibrium.

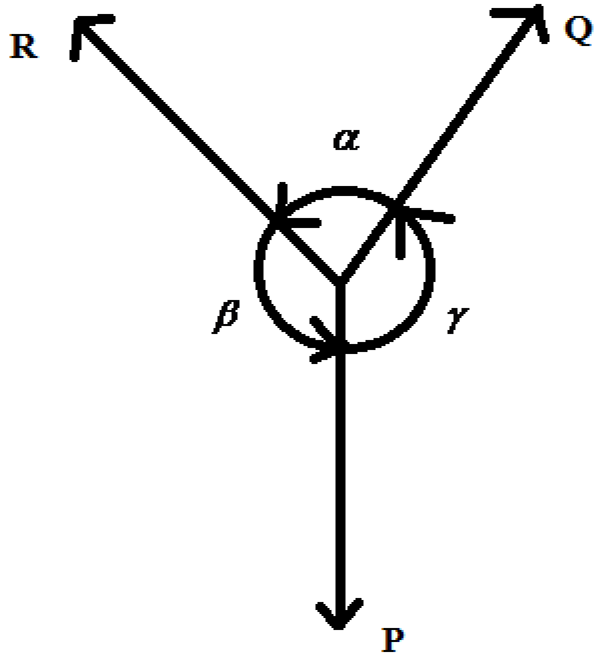
Problem:

State and prove laming theorem.

Statement:

If a particle is in equilibrium under the acting of three forces $\vec{P},\vec{Q},\vec{R}$ then show that $\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}$ where α is the angle between $\vec{Q}$ and $\vec{R}$, β is the angle between $\vec{R}$ and $\vec{P}$ and γ is the angle between $\vec{P}$ and $\vec{Q}$.

Proof:



If a particle is in equilibrium under action of three forces

⇨ The resultant of three forces is zero

$\vec{P}+\vec{Q}+\vec{R}=0$ --------------------------------(1)

Multiply $\vec{P}$ on both sides

$\vec{P}X\vec{P}+\vec{Q}X\vec{P}+\vec{R}X\vec{P}=0$ since $\vec{a}X\vec{b}=|\vec{a}||\vec{b}|\sin\theta x$

Let x be the the unit vector $\perp r$ to three forces

$P^2\cancel{\sin 0}+PQ\sin\gamma\widehat{x+}PR\sin\beta(-x)=0$

$PQ\sin\gamma x-PR\sin\beta x=0$

$PQ\sin\gamma x=PR\sin\beta x$

$\cancel{P}Q\sin\gamma\widehat{x=}\cancel{P}R\sin\beta\bar{x}$

$Q\sin\gamma=R\sin\beta$

$$\frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} \text{---------------------------------------}(2)$$

Taking Cross product by $\vec{Q}$ in (1)

$$\vec{P}X\vec{Q} + \vec{Q}X\vec{Q} + \vec{R}X\vec{Q} = 0$$

$$QP\sin\gamma(-x) + Q^2 \cancel{\sin 0} + QR\sin\alpha\widehat{x} = 0$$

$$-QP\sin\gamma\widehat{x} + QR\sin\alpha\widehat{x} = 0$$

$$QP\sin\gamma\widehat{x} = QR\sin\alpha\widehat{x}$$

$$P\sin\gamma = R\sin\alpha$$

$$\frac{P}{\sin\alpha} = \frac{R}{\sin\gamma} \text{---------------------------------------}(3)$$

From (2) and (3)

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Problem:

Let $\vec{F_1}, \vec{F_2}, \vec{F_3}$ are the three force of a particle

Since the forces keep the particle in equilibrium

$$\vec{F_1} + \vec{F_2} + \vec{F_3} = 0 \text{---}(1)$$

Let x be the unit vector perpendicular to $\vec{F_1}$ *and* $\vec{F_2}$

$\Rightarrow x \bullet \vec{F}_1 = 0$ and $x \bullet \vec{F}_2 = 0$

Taking dot product by $\hat{x}$ with (1)

Hence all the three forces are coplanar

$$x \bullet \left(\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} \right) = 0$$

$$\left(x \bullet \overrightarrow{F_1} + x \bullet \overrightarrow{F_2} + x \bullet \overrightarrow{F_3} \right) = 0$$

$$(0 + 0 + x \bullet \overrightarrow{F_3}) = 0$$

$$x \bullet \overrightarrow{F_3} = 0$$

$\Rightarrow \overrightarrow{F_3}$ is $\perp r$ to x

Hence all the three forces are coplanar.

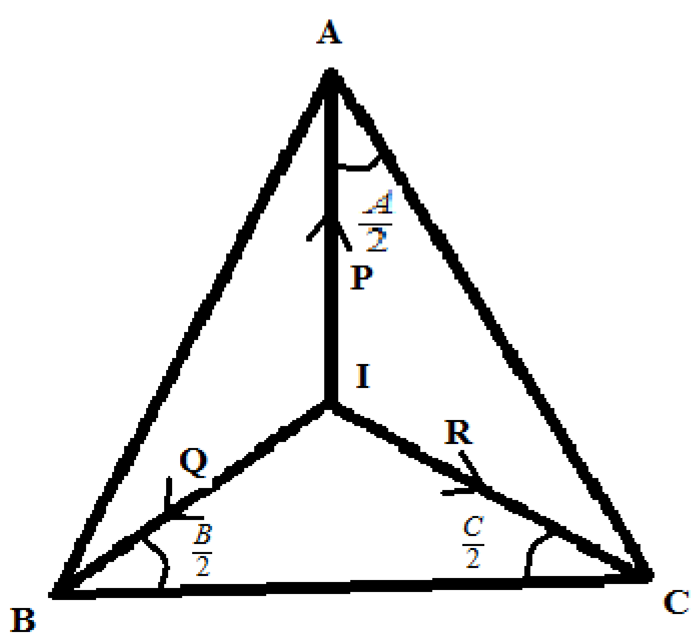
Problem:

If I is the incentre (orthocenter) of a triangle ABC if the forces of the magnitude P,Q,R acting along the bisectors IA, IB, IC are in equilibrium such that $\dfrac{P}{\cos A/2} = \dfrac{Q}{\cos B/2} = \dfrac{R}{\cos C/2}$

Solution:

The forces P, Q, R act at I and in equilibrium

By lami's theorem



$$\frac{P}{\sin BIC}=\frac{Q}{\sin CIA}=\frac{R}{\sin AIB} \text{-----------------------------------}(1)$$

In Tringle BIC

$$\frac{B}{2}+\frac{C}{2}+BIC=180^\circ$$

$$BIC=180^\circ-\left(\frac{B}{2}+\frac{C}{2}\right)$$

$$Sin\,BIC=Sin\left[180^\circ-\left(\frac{B}{2}+\frac{C}{2}\right)\right]$$

$$=\sin\left(\frac{B}{2}+\frac{C}{2}\right) \quad \text{since } \sin(180^\circ-\theta)=\sin\theta$$

$$=\sin\left(90^\circ-\frac{A}{2}\right)$$

$$Sin\,BIC=\cos\left(\frac{A}{2}\right)$$

Similarly

$$\sin CIA=Sin\left[180^\circ-\left(\frac{A}{2}+\frac{C}{2}\right)\right]$$

$$=Sin\left[\left(\frac{A}{2}+\frac{C}{2}\right)\right]$$

$$=Sin\left(90^\circ-\frac{B}{2}\right)$$

$$=\cos\frac{B}{2}$$

And

$$\sin AIB = Sin\left[180^\circ - \left(\frac{A}{2}+\frac{B}{2}\right)\right]$$

$$= Sin\left[\left(\frac{A}{2}+\frac{B}{2}\right)\right]$$

$$= Sin\left(90^\circ - \frac{C}{2}\right)$$

$$= \cos\frac{C}{2}$$

From the above Equation (1) Becomes

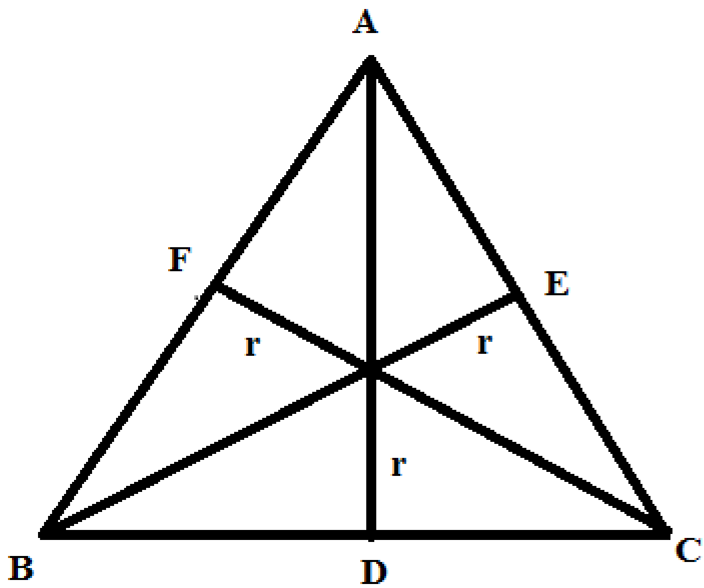
$$\frac{P}{\cos A/2} = \frac{Q}{\cos B/2} = \frac{R}{\cos C/2}$$

Problem:

If I is the incentre of a triangle ABC if the forces IA,IB, IC acting at I are in equilibrium . Such that ABC is an equilateral triangle.

Solution:

The forces IA, IB, IC at I and are in equilibrium.



By Lami's theorem

$$\frac{P}{\sin BIC} = \frac{Q}{\sin CIA} = \frac{R}{\sin AIB}$$

$$\frac{AI}{\cos A/2} = \frac{BI}{\cos B/2} = \frac{CI}{\cos C/2} \text{ -------------------------------(1)}$$

In a triangle AIE

$$Sin\frac{A}{2} = \frac{IE}{AI} = \frac{r}{AI}$$

$$AI = \frac{r}{Sin\frac{A}{2}}$$

In a triangle BIF

$$Sin\frac{B}{2} = \frac{r}{BI}$$

$$BI = \frac{r}{Sin\frac{B}{2}}$$

In a triangle CIF

$$Sin\frac{C}{2} = \frac{r}{CI}$$

$$CI = \frac{r}{Sin\frac{C}{2}}$$

(1)becomes

$$\frac{r}{\sin\frac{A}{2}\cos\frac{B}{2}} = \frac{r}{\sin\frac{B}{2}\cos\frac{B}{2}} = \frac{r}{\sin\frac{C}{2}\cos\frac{B}{2}}$$

$$\frac{r}{\left(\sin\frac{A}{2}\right)} = \frac{r}{\left(\sin\frac{B}{2}\right)} = \frac{r}{\left(\sin\frac{C}{2}\right)}$$

$$\sin 2A = 2\sin A\cos$$

$Put\ \ A = \frac{x}{2}$

$$\frac{\sin x}{2} = \sin\frac{x}{2}\cos x$$

$$\Rightarrow \frac{2r}{\sin A} = \frac{2r}{\sin B} = \frac{2r}{\sin C}$$

$+2r$

$$\Rightarrow \frac{1}{\sin A} = \frac{1}{\sin B} = \frac{1}{\sin C}$$

$$\Rightarrow \sin A = \sin B = \sin C$$

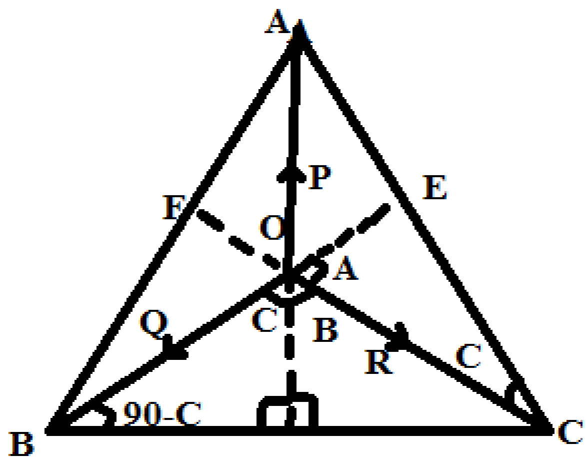
$$\Rightarrow A = B = C$$

Problem:

If 0 is the orthocentre of a triangle if the forces of Magnitude P,Q,R acting along OA,OB,OC arc

in equilibrium . Such that $\frac{p}{a} = \frac{q}{b} = \frac{r}{c}$

Solution:



The Forces P,Q,R act at 0 and are in equilibrium by Lami's theorem

$$\frac{P}{\sin BOC} = \frac{Q}{\sin COA} = \frac{R}{\sin AOB} \text{ ------------------------(1)}$$

If a triangle CBE

$$90 + CBE + C = 180^0$$

$$CBE = 180^0 - 90 - C$$

$$\boxed{CBE = 90^\circ - C}$$

In a triangle BOD

$$90 - C + 90 + BOD = 180^0$$

$$\boxed{BOD = C}$$

In a triangle BCF

$$90 + B + BCF = 180^0$$

$$\boxed{BCF = 90 - B}$$

In a triangle COD

$$90 + 90 - B + COD = 180^0$$

$$\boxed{COD = B}$$

In BOC

$$\sin BOC = Sin(B+C)$$
$$= Sin\left[180^{\circ} - A\right]$$
$$\sin BOC = Sin(\mathrm{A})$$

In

$$Sin\,COA = Sin(C+A)$$
$$= \mathrm{Sin}(180^{\circ} - B)$$
$$= Sin(B)$$

$$Sin\,AOB = Sin(\mathrm{A}+\mathrm{B})$$
$$= \mathrm{Sin}(180^{\circ} - C)$$
$$= Sin(\mathrm{C})$$

(1) Becomes,

$$\frac{P}{\sin A} = \frac{Q}{\sin \mathrm{B}} = \frac{R}{\sin \mathrm{C}}$$

But sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin \mathrm{B}} = \frac{c}{\sin \mathrm{C}}$$

$\sin A : \sin \mathrm{B} : \sin \mathrm{C} = a : \mathrm{b} : \mathrm{c}$

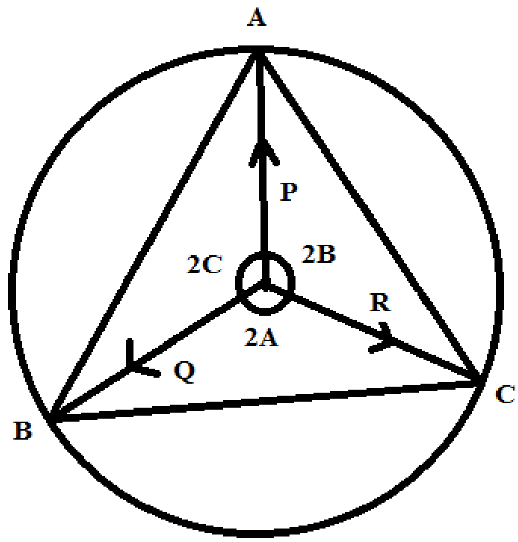
$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$$

Problem: S is the circum centre of a triangle ABC if forces of magnitudes P, Q, R acting along SA, SB, SC are in equilibrium

Such that (i) $\frac{P}{\sin\ 2A} = \frac{Q}{\sin\ 2\mathrm{B}} = \frac{R}{\sin\ 2\mathrm{C}}$

(ii) $\frac{P}{a^2(\mathrm{b}^2+c^2-a^2)} = \frac{Q}{\mathrm{b}^2(\mathrm{c}^2+a^2-b^2)} = \frac{R}{c^2(\mathrm{a}^2+b^2-c^2)}$

Solution:



Part (i)

If the forces P, Q, R act at S are in equilibrium

By Lami's theorem

$$\frac{P}{\sin\ 2A} = \frac{Q}{\sin\ 2\mathrm{B}} = \frac{R}{\sin\ 2\mathrm{C}}$$

Part: (ii)

By sine formula

$$\frac{a}{\sin\ A} = \frac{b}{\sin\ \mathrm{B}} = \frac{c}{\sin\ \mathrm{C}}$$

$$\sin A : \sin \mathrm{B} : \sin \mathrm{C} = a : \mathrm{b} : \mathrm{c}$$

By Cosine Formula

1) $c^2 = a^2 + b^2 - 2ab\cos\ c$

$$\cos \mathrm{c} = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos \mathrm{c} = \frac{c\left(a^2 + b^2 - c^2\right)}{2abc}$$

2) $$a^2 = b^2 + c^2 - 2bc\cos\ A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{a\left(b^2 + c^2 - a^2\right)}{2abc}$$

3) $$b^2 = c^2 + a^2 - 2ca\cos\ B$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\cos B = \frac{b(c^2 + a^2 - b^2)}{2abc}$$

(1) Becomes

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

$$\frac{P}{2\sin A\cos A} = \frac{Q}{2\sin\mathrm{B}\cos\mathrm{B}} = \frac{R}{2\sin\mathrm{C}\cos\mathrm{C}}$$

$$\frac{P}{\frac{\cancel{2}(\mathrm{a})\,\mathrm{a}(\mathrm{b}^2 + c^2 - a^2)}{\cancel{2}abc}} = \frac{Q}{\frac{\cancel{2}(\mathrm{b})\,\mathrm{b}(\mathrm{c}^2 + a^2 - b^2)}{\cancel{2}abc}} = \frac{R}{\frac{\cancel{2}(\mathrm{c})\,\mathrm{c}(\mathrm{a}^2 + b^2 - c^2)}{\cancel{2}abc}}$$

$$\frac{P(abc)}{\mathrm{a}^2(\mathrm{b}^2 + c^2 - a^2)} = \frac{\mathrm{Q}(abc)}{b^2(\mathrm{c}^2 + a^2 - b^2)} = \frac{\mathrm{R}(abc)}{c^2(\mathrm{a}^2 + b^2 - c^2)}$$

$\div\ abc$

$$\frac{P}{\mathrm{a}^2(\mathrm{b}^2 + c^2 - a^2)} = \frac{\mathrm{Q}}{b^2(\mathrm{c}^2 + a^2 - b^2)} = \frac{\mathrm{R}}{c^2(\mathrm{a}^2 + b^2 - c^2)}$$